

Control of Singularly Perturbed Hybrid Stochastic Systems

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Abstract—In this paper, we study a class of optimal stochastic control problems involving two different time scales. The fast mode of the system is represented by deterministic state equations whereas the slow mode of the system corresponds to a jump disturbance process. Under a fundamental “ergodicity” property for a class of “infinitesimal control systems” associated with the fast mode, we show that there exists a limit problem which provides a good approximation to the optimal control of the perturbed system. Both the finite- and infinite-discounted horizon cases are considered. We show how an approximate optimal control law can be constructed from the solution of the limit control problem. In the particular case where the infinitesimal control systems possess the so-called turnpike property, i.e., are characterized by the existence of global attractors, the limit control problem can be given an interpretation related to a decomposition approach.

I. INTRODUCTION

THIS PAPER deals with the approximation of the optimal control of a class of hybrid *piecewise deterministic control systems* (PDCS), where the jump disturbances are state and control dependent, and when the time scales of the stochastic and the deterministic parts are of different orders of magnitude. More precisely, we shall assume that the deterministic state equations defining the evolution of the “continuous” state variable correspond to the *fast mode* of the system, whereas the “discrete” state variable, which evolves according to a stochastic jump process, defines the *slow mode*.

The control of PDCSs has been the object of considerable investigation in Control Theory (see [66], [54], [55], [19], and [63] for a sample of the literature on this topic). Recently, this class of control systems has provided an elegant paradigm for the study of manufacturing systems (see [48], [13], [4], and [56]). Typically, in these models, the stochastic jump process describes the evolution of the operational state of a flexible manufacturing shop, with jumps due to failures and repairs of the machines, whereas the deterministic state equations represent the evolution of the surplus of parts produced by the system. In most of these models, the jump Markov disturbances due to failures and repairs are assumed to be represented as a continuous-homogenous Markov chain with jump rates which are indepen-

dent of state and control. In [15], a model has been proposed where, for each machine of the shop, an additional state variable records the *age* of the machine and the failure rates are age dependent. This model provided an example of a PDCS with state dependent jump rates. In [42], a manufacturing system with control (production rate) dependent failure rates is studied. For related works dealing with manufacturing systems, we refer the reader to [14], [16], [17], [22], [23], [30]–[32], [35], [44], and [57]–[60].

The class of systems we study in this paper corresponds to a situation where a basically deterministic plant (e.g., a production system), called the fast subsystem is subject to infrequent modal disruptions occurring randomly (e.g., the machine failures process), called the slow subsystem. The limit optimal control problem, obtained when the time scale ratio between the slow and the fast processes tends to infinity, is nontrivial as long as the transition probabilities for the perturbing stochastic process depend on the control exercised on the fast system and on its state evolution. In a production system environment, this would be the case if, among the (fast) state variables one has, for example, the temperature or the pressure which not only influences the yield of the process, but also influences the probability of failures. Indeed, this defines an environment which is natural but significantly different from the one considered by Olsder–Suri [48].

The method of approximation of the optimal control proposed in this paper is related to the theory of control of singularly perturbed systems. A traditional approach to the control of singularly perturbed systems is to equate the perturbation parameter to zero and then use the so-called “boundary layer method.” This *reduction technique* stems from the seminal works of Tichonov [61], Vasil’eva and Butuzov [62], O’Malley [46] and proved to be very successful in many applications (see [11], [21], [38]–[40], [47], [50], and [53]). We shall use a different approach here. It is related to the *averaging technique* developed in [5], [6], [8], [9], [25]–[28], [33], [34], [52], and [64]. The technique uses the dynamic programming tenet of transition associated with a change of time scale in a class of locally defined *infinitesimal deterministic control problems*. The technique has mostly been used for singularly perturbed deterministic systems, and the results reported here seem to be its first adaptation to a stochastic control context. We specialize the analysis to a class of singularly perturbed PDCSs that lend themselves nicely to a dynamic programming approach which is well adapted to our averaging technique. In [41], a different averaging technique is proposed for the analysis of singularly perturbed controlled jump-diffusion processes. The very general technique of [41] is based on Martingale theory and weak convergence of proba-

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bility measures. Our method uses more straightforward analysis to derive the approximation error bounds. More importantly, in our approach, we have been able to analyze the case where the fast mode of the system is controlled, which belongs to a notoriously difficult class of problems (cf. [41]).

The paper is organized as follows. In Section II, we define precisely the class of systems under consideration, when the time horizon is finite. In Section III, we prove the convergence to a limit control problem and in Section IV, we show how to define an approximate optimal control for the perturbed system. Section V extends these results to the case of infinite time horizon with discounted cost. Section VI gives an interpretation of the limit control problem in the case where the infinitesimal control problems satisfy the so-called *turnpike property*, i.e., when the optimal piecewise deterministic trajectories have attractors, called the turnpikes.

II. A TWO-TIME-SCALE PIECEWISE DETERMINISTIC CONTROL SYSTEM

We consider a hybrid control system described by two types of state variables. One is a vector of variables continuously changing in \mathbb{R}^p , and the other is a stochastic jump process taking values in a finite index state space I . Corresponding to any state $i \in I$, there exists a system of differential equations describing the dynamics of the continuously changing variables under the condition that the jump process is in the state i . The “continuous” state variables can be associated with the “deterministic” dynamics of a plant while the stochastic jump process represents the changes of its operational modes. A small parameter $\varepsilon > 0$ is introduced below in such a way that continuous variables can have a finite (not tending to zero with ε) deviation on any time interval of the length ε while the probability for the jump process to change its value on such an interval is of the order $O(\varepsilon)$. Thus, continuous variables can be considered to be “fast” with respect to the rate of the occurrence of the jumps.

A. Fast Deterministic System

Assume that a continuous state variable $x \in \mathbb{R}^p$ evolves according to the state equation

$$\varepsilon \frac{dx}{dt} = f^i(x, u) \quad (1)$$

$$u \in U^i \quad (2)$$

where $U^i \subset \mathbb{R}^m$ is a given control constraint set and $f^i(x, u)$ satisfies the usual smoothness conditions for optimal control problems (C^1 in x , continuous in u). This state equation is indexed over $i \in I$. An admissible control for the system (1) and (2) is a measurable function $u(t)$ taking its values in U^i .

It will be convenient to define a “stretched out time scale” via the transformation $\tau = t/\varepsilon$. It will be assumed that, given an initial state x^0 and an admissible control $u(t)$ there exists a unique trajectory $\tilde{x}(\cdot): [0, \infty) \mapsto \mathbb{R}^p$ which is the solution to

$$\frac{d\tilde{x}(\tau)}{d\tau} = f^i(\tilde{x}(\tau), \tilde{u}(\tau)) \quad (3)$$

$$\tilde{u}(\tau) \in U^i \quad (4)$$

$$\tilde{x}(0) = x^0 \quad (5)$$

where we have used the following notations $\tilde{x}(\tau) = x(\varepsilon\tau)$, $\tilde{u}(\tau) = u(\varepsilon\tau)$.

B. Slow Stochastic Jump Process

We assume that a discrete-state variable is “moving slowly” according to a continuous time stochastic jump process with transition rates

$$\begin{aligned} P[\xi(t+\delta) = j | \xi(t) = i, x(t) = x, u(t) = u] \\ = q_{ij}(x, u)\delta + o(\delta), \quad i \neq j, \delta > 0 \end{aligned} \quad (6)$$

$$\lim_{\delta \rightarrow 0} \frac{o(\delta)}{\delta} = 0 \quad (7)$$

where the transition rates $q_{ij}(x, u)$ are continuous functions and the limit in (7) is uniform in x and u inside a sufficiently large domain. Here, as usual, $q_{ij}(\cdot, \cdot) \geq 0$ if $i \neq j$, $q_{ii}(\cdot, \cdot) < 0$, and $\sum_{j \in I} q_{ij}(\cdot, \cdot) = 0$. We also use the notation $q^i(\cdot, \cdot) = \sum_{j \neq i} q_{ij}(\cdot, \cdot) > 0$ for the jump rate of the ξ -process.

C. Admissible Policies and Performance Criterion

Consider any finite partition of the time interval $[0, T] = \bigcup_{k=0}^K [v_k, v_{k+1}]$ where

$$v_0 = 0 < v_1 < \dots < v_K < v_{K+1} = T.$$

A policy will be defined to be a selection of such a partition and a mapping $\gamma: [0, T] \times I \times \mathbb{R}^p \rightarrow \mathcal{U}(t, i)$, where $\mathcal{U}(t, i)$ is the class of measurable mappings $u(\cdot): [t, T] \rightarrow U^i$. That is, $\gamma(t, i, x) = u(\cdot): [t, T] \rightarrow U^i$. Such a policy is implemented recursively (over time) in the following manner.

Let t_n denote the n th decision time and allow it to have the following dual nature: t_n is either an endpoint v_k of the partition, or a random time at which a jump of the ξ -process occurs. At $t_n < T$, the controller observes $s^n = (i^n, x^n) = s(t_n)$, where $i^n = \xi(t_n)$ and $x^n = x(t_n)$, and chooses an admissible control $u^n(\cdot) = \gamma(t_n, i^n, x^n)$ which is a function mapping $[t_n, T] \mapsto U^{i^n}$. The associated trajectory $x^n(\cdot): [t_n, T] \mapsto \mathbb{R}^p$ is the solution of

$$\varepsilon \frac{dx^n(t')}{dt'} = f^{i^n}(x^n(t'), u^n(t')) \quad (8)$$

$$u^n(t') \in U^{i^n} \quad (9)$$

$$x^n(t_n) = x^n. \quad (10)$$

This control and trajectory will be acting until t_{n+1} when either the ξ -process jumps again, or the next end point of an interval of the partition is reached. Of course, no change in control is permitted if the terminal time T is reached before t_{n+1} . A policy is admissible if it defines a measurable random process $\{t_n, s^n\}$, $n = 0, 1, \dots$. The policy so defined allows a change of control not only at the moments of random jumps, but also at the partition points $\{v_k\}$. Notice that by allowing such a change, we do not actually extend the class of policies with respect to one in which the control is allowed to change only at the moments of random jumps. However, a separation of the partition as a part of the policy proves to be more convenient in dealing

with the construction of near optimal controls (see Section IV-B below). Associated with an initial time \bar{t} , state \bar{s} , and an admissible policy γ , we define the following performance criterion, for the time interval $[\bar{t}, T]$

$$J_\varepsilon^\gamma(\bar{t}, \bar{s}) = E_\gamma \left[\int_{\bar{t}}^T L^{\xi(t)}(x(t), u(t)) dt + G(\xi(T)) | s(\bar{t}) = \bar{s} \right] \quad (11)$$

where $G(i)$ is a terminal cost incurred when $\xi(T) = i$, and where $L^i(x, u)$ is a continuous function which gives the rate at which cost accumulates in this system for $i \in I$. Of course, the right-hand side of (11) depends on ε via $x(t)$, which is a solution of (8)–(10). Notice that we assume that this terminal cost does not depend on the value of the “fast” state variable $x(T)$. We are interested in approximating the optimal value function

$$J_\varepsilon^*(t, s) = \inf_\gamma J_\varepsilon^\gamma(t, s) \quad (12)$$

by a suitably constructed limit value function.

Assumption 1: The sets U^i , $i \in I$, are compact, and there exists a compact set $X \subset R^p$ such that for any admissible policy

$$x(t) \in X \quad \forall t \in [0, T]. \quad (13)$$

Note that X is assumed to be independent of ε .

Remark 1: Assumption 1 is used to establish an asymptotic representation of the dynamic programming equations derived in Lemmas 1 and 2. This assumption is easily verified. It is trivially satisfied, for instance, for systems defined on compact manifolds (compare [10] and [29]). It is also satisfied for systems having suitable stability properties (see Remark 3 in Section III-B1). Let us note as well that for some important classes of systems, Assumption 1 can be too restrictive. It is not valid, for example, for linear systems with “unstable” eigenvalues (like ones considered in [4]). To extend the applicability of our approach to these classes of systems, one may replace Assumption 1 by a weaker assumption (however, still allowing one to establish the above mentioned lemmas). It postulates that an arbitrary approximation to the optimal value $J_\varepsilon^*(t, s)$ can be achieved with an admissible policy which generates $x(t)$ satisfying (13). Although this “coercivity type” assumption is common in optimal control theory [9], its verification for the class of hybrid systems we are dealing with can be quite involved and we do not consider it in this paper.

Remark 2: The terminal cost in (11) is assumed to be independent of the value of the fast variable $x(T)$. The general case when it does depend on this value and is defined by a function $F(x(T), \xi(T))$ can, under certain conditions, be reduced to the case under consideration with

$$G(i) = \inf_x F(x, i) \quad (14)$$

where \inf is sought over the set of all x reachable along the trajectories of (3)–(5). We will not give an exact statement justifying such a reduction, but the idea is that on the interval $[T - \varepsilon^{1/2}, T]$, the jump variable is likely to stay fixed and equal to, say, i . In the stretched time scale of system (3)–(5), this interval is of the length $\varepsilon^{-(1/2)}$, and one can use a control steering the x -trajectory to the (near) minimizer in (14). Similar reductions in deterministic optimal control setting was established in [27, Th. 4.4] (see also [7], [10], and [43, p. 271]).

III. CONVERGENCE TO A LIMIT-CONTROL PROBLEM

A. Tenet of Transition

In this subsection, we recall the elementary steps in the dynamic programming approach applied to PDCSs. These results are included here for the sake of completeness, and also because they are important to the convergence proof that follows in order to have a precise estimate of the approximation error that is associated with an infinitesimal time interval Δ .

Lemma 1: Under Assumption 1, for any policy γ admissible on $[\bar{t}, T]$ and for any set of bounded Borel functions $h^i(x, u)$, $i \in I$, the following holds true, for $\bar{t} < T$ and Δ sufficiently small

$$E_\gamma \left[\int_{\bar{t}}^{\bar{t}+\Delta} h^{\xi(t)}(x(t), u(t)) dt | \xi(\bar{t}) = i, x(\bar{t}) = \bar{x} \right] = \int_{\bar{t}}^{\bar{t}+\Delta} h^i(x(t), u(t)) dt + O(\Delta^2) \quad (15)$$

where $(\xi(t), x(t), u(t))$ are generated by the policy γ and the initial conditions $\xi(\bar{t}) = i, x(\bar{t}) = \bar{x}$ in the left hand side and $x(t)$ being the solution of (1) associated with $u(t)$ and $x(\bar{t}) = \bar{x}$ as an initial condition in the right-hand side. $O(\Delta^2)$ is a function of Δ such that $\lim_{\Delta \rightarrow 0} |O(\Delta^2)/\Delta^2| \leq c = \text{constant}$.

Note that (15) states that when the slow variable $\xi(t)$ is in state i , then the expected payoff under an admissible policy is equal to the deterministic payoff along the corresponding deterministic trajectory and control except for a correction of order Δ^2 .

Proof: We shall use the notation $\bar{s} = (i, \bar{x})$. We consider the random time T of the next jump for the process $\xi(\cdot)$. We have

$$\begin{aligned} E_\gamma \left[\int_{\bar{t}}^{\bar{t}+\Delta} h^{\xi(t)}(x(t), u(t)) dt | s(\bar{t}) = \bar{s} \right] &= E_\gamma \left[\int_{\bar{t}}^{\bar{t}+\Delta} h^{\xi(t)}(x(t), u(t)) dt | s(\bar{t}) = \bar{s}, T > \bar{t} + \Delta \right] \\ &\quad \cdot P_\gamma[T > \bar{t} + \Delta | s(\bar{t}) = \bar{s}] \\ &\quad + E_\gamma \left[\int_{\bar{t}}^{\bar{t}+\Delta} h^{\xi(t)}(x(t), u(t)) dt | s(\bar{t}) = \bar{s}, T \leq \bar{t} + \Delta \right] \\ &\quad \cdot P_\gamma[T \leq \bar{t} + \Delta | s(\bar{t}) = \bar{s}]. \end{aligned} \quad (16)$$

Expressing more precisely the probability of no jump between \bar{t} and $\bar{t} + \Delta$, and using the fact that $\xi(t) \equiv i$ if there is no jump, we obtain

$$\begin{aligned} E_\gamma \left[\int_{\bar{t}}^{\bar{t}+\Delta} h^{\xi(t)}(x(t), u(t)) dt | s(\bar{t}) = \bar{s} \right] \\ = \left\{ \int_{\bar{t}}^{\bar{t}+\Delta} h^i(x(t), u(t)) dt \right\} \\ \cdot \exp \left(- \int_{\bar{t}}^{\bar{t}+\Delta} q^i(x(t), u(t)) dt \right) \\ + \left\{ E_\gamma \left[\int_{\bar{t}}^{\bar{t}+\Delta} h^{\xi(t)}(x(t), u(t)) dt | s(\bar{t}) = \bar{s}, \right. \right. \\ \left. \left. \mathcal{T} \leq \bar{t} + \Delta \right] \right\} \\ \cdot \left(1 - \exp \left(- \int_{\bar{t}}^{\bar{t}+\Delta} q^i(x(t), u(t)) dt \right) \right). \quad (17) \end{aligned}$$

We now use the fact that

$$\begin{aligned} \exp \left(- \int_{\bar{t}}^{\bar{t}+\Delta} q^i(x(t), u(t)) dt \right) \\ = 1 - \int_{\bar{t}}^{\bar{t}+\Delta} q^i(x(t), u(t)) dt + O(\Delta^2) \end{aligned}$$

and the boundedness of $h^i(x, u)$ to obtain the desired result. •

Lemma 2: Let Assumption 1 hold and $\bar{t} \in [0, T)$, $\bar{x} \in X$. Let γ be an admissible policy on $[\bar{t}, T]$. Then,

$$\begin{aligned} J'_\varepsilon(\bar{t}, i, \bar{x}) = J'_\varepsilon(\bar{t} + \Delta, i, x(\bar{t} + \Delta)) + \int_{\bar{t}}^{\bar{t}+\Delta} \\ \cdot \left[L^i(x(t), u(t)) + \sum_{j \in I} q_{ij}(x(t), u(t)) \right. \\ \left. \cdot J'_\varepsilon(\bar{t} + \Delta, j, x(\bar{t} + \Delta)) \right] dt + O(\Delta^2) \end{aligned} \quad (18)$$

where $x(t)$ is as in the right-hand side of (15).

Proof: By definition, the following holds:

$$\begin{aligned} J'_\varepsilon(\bar{t}, i, \bar{x}) = E_\gamma \left[\int_{\bar{t}}^{\bar{t}+\Delta} L^{\xi(t)}(x(t), u(t)) dt | s(\bar{t}) = \bar{s} \right] \\ + \sum_{j \in I} P_\gamma [\xi(\bar{t} + \Delta) = j | \xi(\bar{t}) = i] \\ \cdot J'_\varepsilon(\bar{t} + \Delta, j, x(\bar{t} + \Delta)). \quad (19) \end{aligned}$$

Now, using Lemma 1 for the $L^i(x, u)$ functions, we obtain

$$\begin{aligned} J'_\varepsilon(\bar{t}, i, \bar{x}) = \int_{\bar{t}}^{\bar{t}+\Delta} L^i(x(t), u(t)) dt \\ + \sum_{j \in I} P[\xi(\bar{t} + \Delta) = j | \xi(\bar{t}) = i] \\ \cdot J'_\varepsilon(\bar{t} + \Delta, j, x(\bar{t} + \Delta)) + O(\Delta^2). \quad (20) \end{aligned}$$

By the same argument as already used in the proof Lemma 1, we obtain

$$\begin{aligned} P[\xi(\bar{t} + \Delta) = j | \xi(\bar{t}) = i] \\ = \delta_{ij} + \int_{\bar{t}}^{\bar{t}+\Delta} q_{ij}(x(t), u(t)) dt + O(\Delta^2) \quad (21) \end{aligned}$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

The result (18) is then obtained from (20) and (21). •

Corollary 1: If Assumption 1 holds, the following is true for any $\bar{t} \in [0, T)$, \bar{x} and, sufficiently small $\Delta > 0$,

$$\begin{aligned} J^*_\varepsilon(\bar{t}, i, \bar{x}) \\ = \inf_{u(\cdot)} \left\{ J^*_\varepsilon(\bar{t} + \Delta, i, x(\bar{t} + \Delta)) + \int_{\bar{t}}^{\bar{t}+\Delta} \right. \\ \cdot \left[L^i(x(t), u(t)) + \sum_{j \in I} q_{ij}(x(t), u(t)) \right. \\ \left. \cdot J^*_\varepsilon(\bar{t} + \Delta, j, x(\bar{t} + \Delta)) \right] dt \Big\} + O(\Delta^2). \quad (22) \end{aligned}$$

B. Convergence of the Value Function

In this subsection, we introduce the core objects of our approach which are a family of infinitesimal control problems and a system of differential equations for the limit value function.

1) *An Associated Class of Infinitesimal Deterministic Control Problems:* For any vector $\mathbf{v} = \{v(j)\}_{j \in I}$, consider the family of deterministic optimal control problems

$$\begin{aligned} H^i(\theta, x^0, \mathbf{v}) = \inf \frac{1}{\theta} \int_0^\theta \left(L^i(\tilde{x}(\tau), \tilde{u}(\tau)) \right. \\ \left. + \sum_{j \in I} q_{ij}(\tilde{x}(\tau), \tilde{u}(\tau))v(j) \right) d\tau \quad (23) \end{aligned}$$

s.t.

$$\frac{d\tilde{x}(\tau)}{d\tau} = f^i(\tilde{x}(\tau), \tilde{u}(\tau)) \quad (24)$$

$$\tilde{u}(\tau) \in U^i \quad (25)$$

$$\tilde{x}(0) = x^0. \quad (26)$$

These problems, defined over the stretched out time scale, will be called the *infinitesimal control problems*.¹ The term *infinitesimal* emphasizes the fact that, in the fast time scale, we shall have an essentially infinite horizon control problem defined locally for almost every intermediate time $\bar{t} \in [0, T]$.

Assumption 2: There exist two constants $A > 0$ and $0 < \alpha \leq 1$, and for each i a function $H^i(\mathbf{v})$, such that for all $i \in I$, $x^0 \in X$, and \mathbf{v} in some bounded set Ω

$$|H^i(\theta, x^0, \mathbf{v}) - H^i(\mathbf{v})| \leq \frac{A}{\theta^\alpha}. \quad (27)$$

¹The term infinitesimal control problem has been coined by Z. Artstein.

Remark 3: Assumption 2 is satisfied if trajectories of (24) obtained with the same control satisfy

$$\|x_1(\tau) - x_2(\tau)\| \leq \xi(\tau)\|x_1(0) - x_2(0)\| \quad (28)$$

with $\xi(\tau) \rightarrow 0$ and $\int_0^\infty \xi(\tau) d\tau < \infty$ [see [27, Th. 4.1], and also results in [25] and [26]; in this case, $\alpha = 1/2$ in (27)]. The estimate (28) is valid, for instance, if

$$f^i(x, u) = A^i x + B^i u \quad (29)$$

and eigenvalues of A^i have negative real parts (see [27, Example 3.1]). A more general condition for (28) to take place is that there exist positive definite matrices P and Q , such that for any $u \in V^i$ and any x', x''

$$\begin{aligned} (f^i(x', u) - f^i(x'', u))^T P(x' - x'') \\ \leq -(x' - x'')^T Q(x' - x'') \end{aligned} \quad (30)$$

(this condition was introduced in [21] for $P = I$ and $Q = \beta I$, where I is the identity matrix and β is a positive constant). It can be shown that (30) is also a sufficient condition for Assumption 1 to be satisfied in case the sets

$$X^i \stackrel{\text{def}}{=} \{x | f^i(x, u) = 0, \text{ for some } u \in U^i\}$$

have a nonempty intersection.

Condition (28) is of a stability type. Assumption 2 is also implied by a controllability type condition ([26, Th. 1.3.1] and [33, Cor. 4.1]) postulating that any two points in X can be connected by a trajectory of (24) obtained with some admissible control, the time required for the transition along such trajectory being bounded by some given constant. Notice that this condition is most efficient for systems defined on compact manifolds (see an example in [29]) and some sufficient conditions for the specified type of controllability in [10].

Remark 4: Assumption 2 clearly resembles an ergodicity property. When the time horizon θ goes to ∞ , the optimal value becomes independent of the initial state. Such a property is expected if the system admits an optimal steady state which is a common attractor for all optimal trajectories. This has been called the turnpike property and Section VI will provide more details concerning this case. Another possibility for observing such an ergodic behavior is to obtain a periodic control when $\theta \rightarrow \infty$.

Remark 5: As $H^i(\theta, x^0, \mathbf{v})$ is the value function of a control problem depending on the parameter \mathbf{v} in a linear way, and the bound in (27) is uniform, the “limit” $H^i(\mathbf{v})$ of $H^i(\theta, x^0, \mathbf{v})$ is a Lipschitz function.

2) *The Limit Value Function:* Consider the set of coupled differential equations

$$\frac{dJ_0(t, i)}{dt} = -H^i(\mathbf{J}_0(t)), \quad i \in I \quad (31)$$

with terminal conditions

$$J_0(T, i) = G(i) \quad i \in I \quad (32)$$

where we have denoted by $\mathbf{J}_0(t)$ the vector $\{J_0(t, j)\}_{j \in I}$.

Assumption 3: The system (31) and (32) admits a solution which satisfies

$$\mathbf{J}_0(t) \in \Omega, \quad \forall t \in [0, T].$$

Remark 6: As a consequence of (31), for any $t \in [0, T]$ and Δ sufficiently small, the following holds:

$$J_0(t - \Delta, i) - J_0(t, i) = H^i(\mathbf{J}_0(t))(\Delta + O(\Delta^2)). \quad (33)$$

Theorem 1: There exists a constant C such that

$$\begin{aligned} |J_\varepsilon^*(t, i, x) - J_0(t, i)| \\ \leq C \varepsilon^{\alpha/(1+\alpha)} \quad \forall i \in I, t \in [0, T], x \in X. \end{aligned} \quad (34)$$

Proof: Define $\Delta(\varepsilon) \rightarrow 0$ with $\Delta(\varepsilon)/\varepsilon \rightarrow \infty$ when $\varepsilon \rightarrow 0$. For a given $t \in [0, T]$, define $t_\ell = t + \ell\Delta(\varepsilon)$ for $\ell = 0, 1, \dots, \lceil (T-t)/\Delta(\varepsilon) \rceil = L(\varepsilon)$, $t_{L(\varepsilon)+1} = T$, where $\lceil \beta \rceil$ denotes the integer part of the number β . Notice that we may very well have $t_{L(\varepsilon)} = t_{L(\varepsilon)+1}$. By definition, we have

$$J_\varepsilon^*(t_{L(\varepsilon)+1}, i, x) = G(i) \quad (35)$$

$$J_0(t_{L(\varepsilon)+1}, i) = G(i) \quad (36)$$

so we obtain

$$J_\varepsilon^*(t_{L(\varepsilon)+1}, i, x) = J_0(t_{L(\varepsilon)+1}, i) \quad \forall i \in I, \forall x \in X. \quad (37)$$

The proof shall proceed by induction. Assume that

$$\begin{aligned} |J_\varepsilon^*(t_{L(\varepsilon)+1-k}, i, x) - J_0(t_{L(\varepsilon)+1-k}, i)| \\ \leq kD\Delta(\varepsilon) \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right) \quad \forall i \in I, x \in X, \end{aligned} \quad (38)$$

where D is a constant to be specified later and α is the constant introduced in Assumption 2. Notice that $t_{L(\varepsilon)+1-(k+1)} + \Delta(\varepsilon) = t_{L(\varepsilon)+1-k}$. It will be convenient to use the notation $\bar{t}_\varepsilon = t_{L(\varepsilon)+1-(k+1)}$. Using Lemma 2 and Corollary 1, we can write

$$\begin{aligned} \left| J_\varepsilon^*(\bar{t}_\varepsilon, i, x) - \inf_{u(\cdot)} \left\{ J_\varepsilon^*(\bar{t}_\varepsilon + \Delta(\varepsilon), i, x(\bar{t}_\varepsilon) + \Delta(\varepsilon)) \right. \right. \\ \left. \left. + \int_{\bar{t}_\varepsilon}^{\bar{t}_\varepsilon + \Delta(\varepsilon)} \left[L^i(x(t), u(t)) + \sum_{j \in I} J_\varepsilon^*(\bar{t}_\varepsilon + \Delta(\varepsilon), j, x(\bar{t}_\varepsilon) + \Delta(\varepsilon)) q_{ij}(x(t), u(t)) \right] dt \right\} \right| \\ \leq M_1 \Delta^2(\varepsilon) \end{aligned} \quad (39)$$

where M_1 is a positive constant. Using (38), we may then rewrite (39) as

$$\begin{aligned} \left| J_\varepsilon^*(\bar{t}_\varepsilon, i, x) - J_0(\bar{t}_\varepsilon + \Delta(\varepsilon), i) - \inf_{u(\cdot)} \int_{\bar{t}_\varepsilon}^{\bar{t}_\varepsilon + \Delta(\varepsilon)} \right. \\ \left. \cdot \left[L^i(x(t), u(t)) + \sum_{j \in I} J_0(\bar{t}_\varepsilon + \Delta(\varepsilon), j) \right. \right. \\ \left. \left. \cdot q_{ij}(x(t), u(t)) \right] dt \right| \\ \leq kD\Delta(\varepsilon) \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right) + M_1 \Delta^2(\varepsilon). \end{aligned} \quad (40)$$

Consider the integral terms in (40)

$$\int_{\bar{t}_\varepsilon}^{\bar{t}_\varepsilon + \Delta(\varepsilon)} \left[L^i(x(t), u(t)) + \sum_{j \in I} J_0(\bar{t}_\varepsilon + \Delta(\varepsilon), j) \cdot q_{ij}(x(t), u(t)) \right] dt. \quad (41)$$

If we use the stretched out time scale $\tau = t/\varepsilon$, the above integral (41) can be rewritten

$$\Delta(\varepsilon) \left\{ \frac{\varepsilon}{\Delta(\varepsilon)} \int_{\bar{t}_\varepsilon}^{\bar{t}_\varepsilon + (\Delta(\varepsilon)/\varepsilon)} \left[L^i(\tilde{x}(\tau), \tilde{u}(\tau)) + \sum_{j \in I} J_0(\bar{t}_\varepsilon + \Delta(\varepsilon), j) q_{ij}(\tilde{x}(\tau), \tilde{u}(\tau)) \right] d\tau \right\} \quad (42)$$

where $\tilde{x}(\tau)$ and $\tilde{u}(\tau)$ are linked through the state equation (3), with initial condition $\tilde{x}(\bar{t}_\varepsilon) = x$. It is easy to see that, by definition of the infinitesimal problem (23), the following holds:

$$\begin{aligned} \Delta(\varepsilon) \inf_{\tilde{u}(\cdot)} \left\{ \frac{\varepsilon}{\Delta(\varepsilon)} \int_{\bar{t}_\varepsilon}^{\bar{t}_\varepsilon + (\Delta(\varepsilon)/\varepsilon)} \left[L^i(\tilde{x}(t), \tilde{u}(t)) + \sum_{j \in I} J_0^*(\bar{t}_\varepsilon + \Delta(\varepsilon), j) q_{ij}(\tilde{x}(t), \tilde{u}(t)) \right] dt \right\} \\ = \Delta(\varepsilon) H^i \left(\frac{\Delta(\varepsilon)}{\varepsilon}, x, \mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon)) \right). \end{aligned} \quad (43)$$

By Assumption 2, we also have

$$\begin{aligned} \Delta(\varepsilon) \left| H^i \left(\frac{\Delta(\varepsilon)}{\varepsilon}, x, \mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon)) \right) - H^i(\mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon))) \right| \\ \leq \Delta(\varepsilon) \frac{A}{(\Delta(\varepsilon)/\varepsilon)^\alpha}. \end{aligned} \quad (44)$$

Substituting (42)–(44) into (40), we obtain

$$\begin{aligned} |J_\varepsilon^*(\bar{t}_\varepsilon, i, x) - J_0(\bar{t}_\varepsilon + \Delta(\varepsilon), i) - \Delta(\varepsilon) H^i(\mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon)))| \\ \leq kD \Delta(\varepsilon) \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right) \\ + M_1 \Delta^2(\varepsilon) + A \Delta(\varepsilon) \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha. \end{aligned} \quad (45)$$

Notice that from (33), the following holds for some positive constant M_2 :

$$\begin{aligned} |J_0(\bar{t}_\varepsilon, i) - J_0(\bar{t}_\varepsilon + \Delta(\varepsilon), i) - \Delta(\varepsilon) H^i(\mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon)))| \\ \leq M_2 \Delta^2(\varepsilon) \end{aligned} \quad (46)$$

which, along with (45) and after returning to the explicit notation for time subintervals, gives

$$\begin{aligned} |J_\varepsilon^*(t_{L(\varepsilon)+1-(k+1)}, i, x) - J_0(t_{L(\varepsilon)+1-(k+1)}, i)| \\ \leq (k+1)D \Delta(\varepsilon) \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right) \end{aligned} \quad (47)$$

where $D = \max\{M_1 + M_2, A\}$. This establishes the desired induction. Let us take $k = L(\varepsilon) + 1$, so we have

$$\begin{aligned} |J_\varepsilon^*(t) - J_0(t)| &\leq \left\lceil \frac{T-t}{\Delta(\varepsilon)} \right\rceil \Delta(\varepsilon) D \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right) \\ &\leq (T-t + \Delta(\varepsilon)) D \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right) \\ &\leq (T+1) D \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right). \end{aligned} \quad (48)$$

Now, an appropriate selection of $\Delta(\varepsilon)$, for instance $\Delta(\varepsilon) := \varepsilon^{\alpha/(1+\alpha)}$ which satisfies $\lim_{\varepsilon \rightarrow 0} (\Delta(\varepsilon)/\varepsilon) = \infty$, yields the desired result. •

IV. APPROXIMATE OPTIMAL CONTROL

In this section, we again use the averaging technique to show that, once the limit problem is solved, it is possible to construct from its solution an approximate control of the perturbed problem.

A. Near Optimal Control for the Associated Control Problems

Let $\mathbf{v} \in \Omega$, $i \in I$, and $\tilde{u}_{\mathbf{v}}^i(\tau) \in U^i$ be such that

$$\left| \frac{1}{\theta} \int_0^\theta h_{\mathbf{v}}^i(\tilde{x}(\tau), \tilde{u}_{\mathbf{v}}^i(\tau)) d\tau - H^i(\theta, x^0, \mathbf{v}) \right| \leq \frac{A}{\theta^\alpha} \quad (49)$$

where A and α are as in Assumption 2, $\tilde{x}(\tau)$ is the solution of

$$\frac{d\tilde{x}(\tau)}{d\tau} = f^i(\tilde{x}(\tau), \tilde{u}_{\mathbf{v}}^i(\tau)) \quad (50)$$

$$\tilde{x}(0) = x^0 \quad (51)$$

and where we have used the following notation:

$$h_{\mathbf{v}}^i(x, u) = L^i(x, u) + \sum_{j \in I} q_{ij}(x, u) v(j). \quad (52)$$

Notice that the control $\tilde{u}_{\mathbf{v}}^i(\tau)$ providing the fulfillment of (49) always exists.

B. Control Implementation

Let t_ℓ be defined as in the proof of Theorem 1, with $t_0 = 0$, and $t_\ell = \ell \Delta(\varepsilon)$, $\ell = 0, 1, \dots, \lceil T/\Delta(\varepsilon) \rceil = L(\varepsilon)$. On each subinterval $[t_\ell, t_{\ell+1})$, the control implemented will be

$$\hat{u}_\ell(\xi, t) = \tilde{u}_{\mathbf{v}_{\ell+1}^0}^\xi \left(\frac{t - t_\ell}{\varepsilon} \right) \in U^\xi, \quad t \in [t_\ell, t_{\ell+1}), \xi \in I \quad (53)$$

where $\mathbf{v}_{\ell+1}^0 = \mathbf{J}_0(t_{\ell+1})$. We shall denote by $J_\varepsilon^{\hat{u}}(0, i, x)$ the expected cost associated with the use of the above defined control law, with initial conditions $x(0) = x$ and $\xi(0) = i$.

Notice that this control law gives rise to an admissible policy for the PDCS in the sense of Section II-C.

C. Approximation of the Optimal Value Function

Theorem 2: Under Assumptions 1–3, the following inequality holds:

$$|J_\varepsilon^{\hat{u}}(0, i, x) - J_\varepsilon^*(0, i, x)| \leq C \varepsilon^{\alpha/(1+\alpha)}. \quad (54)$$

Proof: According to Theorem 1, it suffices to show that

$$|J_\varepsilon^{\hat{u}}(0, i, x) - J_0(0, i)| \leq C \varepsilon^{\alpha/(1+\alpha)}. \quad (55)$$

By definition, we have

$$J_\varepsilon^{\hat{u}}(T, i, x) = J_0(T, i) \quad (56)$$

since both functions are equal to $G(i)$ when $t = T$. Similarly to (38), assume that

$$\begin{aligned} |J_\varepsilon^{\hat{u}}(t_L(\varepsilon) + 1 - k, i, x) - J_0(t_L(\varepsilon) + 1 - k, i)| \\ \leq k D \Delta(\varepsilon) \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right) \end{aligned} \quad (57)$$

where D will be specified later on. Let us again use the notation $\bar{t}_\varepsilon = t_{L(\varepsilon)+1-(k+1)}$. By Lemma 2, the following holds:

$$\begin{aligned} & \left| J_\varepsilon^{\hat{u}}(\bar{t}_\varepsilon, i, x) - J_\varepsilon^{\hat{u}}(\bar{t}_\varepsilon + \Delta(\varepsilon), i, x(\bar{t}_\varepsilon + \Delta(\varepsilon))) \right. \\ & \quad - \int_{\bar{t}_\varepsilon}^{\bar{t}_\varepsilon + \Delta(\varepsilon)} \left[L^i(x(t), u(t)) + \sum_{j \in I} q_{ij}(x(t), u(t)) \right. \\ & \quad \cdot J_\varepsilon^{\hat{u}}(\bar{t}_\varepsilon + \Delta(\varepsilon), j, x(\bar{t}_\varepsilon + \Delta(\varepsilon))) \left. \right] dt \left. \right| \leq M_1 \Delta^2(\varepsilon) \end{aligned} \quad (58)$$

where $u(t)$ is the control induced by (53) on the interval $[\bar{t}_\varepsilon, \bar{t}_\varepsilon + \Delta(\varepsilon)]$. Taking into account (57) and using the notation of (52), we obtain

$$\begin{aligned} & \left| J_\varepsilon^{\hat{u}}(\bar{t}_\varepsilon, i, x) - J_0(\bar{t}_\varepsilon + \Delta(\varepsilon), i) \right. \\ & \quad - \int_{\bar{t}_\varepsilon}^{\bar{t}_\varepsilon + \Delta(\varepsilon)} h_{\mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon))}^i(x(t), u(t)) dt \left. \right| \\ & \leq M_1 \Delta^2(\varepsilon) + k D \Delta(\varepsilon) \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right). \end{aligned} \quad (59)$$

Changing the time scale to $\tau = (t - t_\ell)/\varepsilon$ in the integral part of (59), and by definition of the feedback \hat{u} , we have

$$\begin{aligned} & \int_{\bar{t}_\varepsilon}^{\bar{t}_\varepsilon + \Delta(\varepsilon)} h_{\mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon))}^i(x(t), u(t)) dt \\ & = \Delta(\varepsilon) \left\{ \frac{\varepsilon}{\Delta(\varepsilon)} \int_0^{\Delta(\varepsilon)/\varepsilon} h_{\mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon))}^i(\tilde{x}(\tau), \tilde{u}_{\mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon))}^i(\tau)) d\tau \right\} \end{aligned}$$

and

$$\begin{aligned} & \left| \Delta(\varepsilon) \left\{ \frac{\varepsilon}{\Delta(\varepsilon)} \int_0^{\Delta(\varepsilon)/\varepsilon} h_{\mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon))}^i(\tilde{x}(\tau), \tilde{u}_{\mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon))}^i(\tau)) d\tau \right\} \right. \\ & \quad \left. - \Delta(\varepsilon) \left\{ H^i \left(\frac{\Delta(\varepsilon)}{\varepsilon}, x, \mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon)) \right) \right\} \right| \\ & \leq A \Delta(\varepsilon) \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha. \end{aligned} \quad (60)$$

By Assumption 2, on the other hand, we may write

$$\begin{aligned} & \left| H^i \left(\frac{\varepsilon}{\Delta(\varepsilon)}, x, \mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon)) \right) - H^i(\mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon))) \right| \\ & \leq A \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha. \end{aligned} \quad (61)$$

Substituting (60) and (61) in (59), we obtain

$$\begin{aligned} & |J_\varepsilon^{\hat{u}}(\bar{t}_\varepsilon, i, x) - J_0(\bar{t}_\varepsilon + \Delta(\varepsilon), i) - \Delta(\varepsilon) H^i(\mathbf{J}_0(\bar{t}_\varepsilon + \Delta(\varepsilon)))| \\ & \leq M_1 \Delta^2(\varepsilon) + k D \Delta(\varepsilon) \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right) \\ & \quad + 2A \Delta(\varepsilon) \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha. \end{aligned} \quad (62)$$

Now, using (46), we obtain a result similar to (47)

$$\begin{aligned} & |J_\varepsilon^{\hat{u}}(\bar{t}_\varepsilon, i, x) - J_0(\bar{t}_\varepsilon, i)| \\ & \leq (k+1) D \Delta(\varepsilon) \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right) \end{aligned} \quad (63)$$

with $D = \max\{M_1 + M_2, 2A\}$. This establishes the induction. Finally, taking $k = L(\varepsilon) + 1$, we obtain

$$\begin{aligned} & |J_\varepsilon^{\hat{u}}(0, i, x) - J_0(0, i)| \\ & \leq \left\lceil \frac{T}{\Delta(\varepsilon)} \right\rceil \Delta(\varepsilon) D \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right) \\ & \leq (T + \Delta(\varepsilon)) D \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right) \\ & \leq (T + 1) D \left(\Delta(\varepsilon) + \left(\frac{\varepsilon}{\Delta(\varepsilon)} \right)^\alpha \right). \end{aligned} \quad (64)$$

To complete the proof, it now suffices to take $\Delta(\varepsilon) = \varepsilon^{\alpha/(1+\alpha)}$. •

V. INFINITE HORIZON WITH DISCOUNTED COST

In the next two sections, we extend the analysis to the case of an infinite horizon-control process with discounted integral cost.

A. Performance Criterion

We consider the same system as in Section II, with a terminal time $T \rightarrow \infty$. A control policy γ is still defined as in Section II-C, with the obvious replacement of T with ∞ . Associated with an admissible policy, we define the following performance criterion:

$$J_\varepsilon^\gamma(\bar{t}, \bar{s}) = E_\gamma \left[\int_{\bar{t}}^\infty e^{-\rho t} L^{\xi(t)}(x(t), u(t)) dt | s(\bar{t}) = \bar{s} \right] \quad (65)$$

where $\rho > 0$ is a given discount rate. We are interested in the optimal value function

$$J_\varepsilon^*(t, s) = \inf_\gamma J_\varepsilon^\gamma(t, s). \quad (66)$$

As usual, when dealing with discounted cost criterion, we shall use the *current-value* cost-to-go value function

$$V_\varepsilon^*(s) = J_\varepsilon^*(0, s) = e^{\rho t} J_\varepsilon^*(t, s). \quad (67)$$

B. Limit Value Functions

Adapting in an obvious way the result obtained in Corollary 1 of Section III, we can write

$$\begin{aligned} V_\varepsilon^*(i, x) &= \inf_{u(\cdot)} \left\{ e^{-\rho\Delta} V_\varepsilon^*(i, x(\Delta)) + \int_0^\Delta e^{-\rho t} \left[L^i(x(t), u(t)) \right. \right. \\ &\quad \left. \left. + \sum_{j \in I} q_{ij}(x(t), u(t)) V_\varepsilon^*(j, x(\Delta)) \right] dt \right\} + O(\Delta^2) \\ &= \inf_{u(\cdot)} \left\{ (1 - \rho\Delta) V_\varepsilon^*(i, x(\Delta)) \right. \\ &\quad \left. + \int_0^\Delta h_{V_\varepsilon^*(x(\Delta))}^i(x(t), u(t)) dt \right\} + O(\Delta^2) \quad (68) \end{aligned}$$

where we have used the fact that $e^{-\rho\Delta} = (1 - \rho\Delta) + O(\Delta^2)$ and the notation

$$h_{V_\varepsilon^*(x, u)}^i(x, u) = L^i(x, u) + \sum_{j \in I} q_{ij}(x, u) v(j)$$

introduced in (52).

Theorem 3: Let Assumptions 1 and 2 be satisfied and the algebraic equation

$$\rho V_0(i) = H^i(\mathbf{V}_0) \quad i \in I \quad (69)$$

have solution $\mathbf{V}_0 = \{V_0(j)\}_{j \in I} \in \Omega$. Then, there exists a constant C such that

$$|V_\varepsilon^*(i, x) - V_0(i)| \leq C \varepsilon^{\alpha/(1+\alpha)} \quad \forall i \in I, x \in X. \quad (70)$$

Proof: Let us consider, for each $i \in I$, the error function $E(i, x)$ defined as

$$E(i, x) = V_\varepsilon^*(i, x) - V_0(i) \quad \forall x \in X.$$

We want to show that

$$\hat{E} = \sup_{i \in I, x \in X} |E(i, x)| \leq C \varepsilon^{\alpha/(1+\alpha)}.$$

For that purpose, we shall use (68) to obtain the following estimate

$$\begin{aligned} V_\varepsilon^*(i, x) &= \inf_{u(\cdot)} \left\{ (1 - \rho\Delta)(V_0(i) + E(i, x(\Delta))) \right. \\ &\quad \left. + \int_0^\Delta \left[h_{V_0}^i(x(t), u(t)) + \sum_{j \in I} q_{ij}(x(t), u(t)) \right. \right. \\ &\quad \left. \left. \cdot E(j, x(\Delta)) \right] dt \right\} + O(\Delta^2) \\ &= \inf_{u(\cdot)} \left\{ (1 - \rho\Delta)V_0(i) + \int_0^\Delta h_{V_0}^i(x(t), u(t)) dt \right. \\ &\quad \left. + (1 - \rho\Delta)E(i, x(\Delta)) + \int_0^\Delta \sum_{j \in I} q_{ij}(x(t), u(t)) \right. \\ &\quad \left. \cdot E(j, x(\Delta)) dt \right\} + O(\Delta^2). \quad (71) \end{aligned}$$

Focusing on the error term

$$\begin{aligned} \phi &= (1 - \rho\Delta)E(i, x(\Delta)) \\ &\quad + \int_0^\Delta \sum_{j \in I} q_{ij}(x(t), u(t)) E(j, x(\Delta)) dt \end{aligned} \quad (72)$$

we see that the following holds:

$$\begin{aligned} \phi &= E(i, x(\Delta)) \left(1 - \rho\Delta + \int_0^\Delta q_{ii}(x(t), u(t)) dt \right) \\ &\quad + \int_0^\Delta \sum_{j \neq i} E(j, x(\Delta)) q_{ij}(x(t), u(t)) dt. \end{aligned} \quad (73)$$

If Δ is sufficiently small, the term multiplying $E(i, x(\Delta))$ is positive, and so are the rates $q_{ij}(x(t), u(t))$ when $j \neq i$. This allows us to write

$$\begin{aligned} |\phi| &\leq \hat{E} \left(1 - \rho\Delta + \int_0^\Delta \sum_{j \in I} q_{ij}(x(t), u(t)) dt \right) \\ &= \hat{E}(1 - \rho\Delta). \end{aligned} \quad (74)$$

This leads, with the help of (23) and (71), to the following evaluation:

$$\begin{aligned} V_\varepsilon^*(i, x) - V_0(i) &= -\rho\Delta V_0(i) + \Delta H^i\left(\frac{\Delta}{\varepsilon}, x, \mathbf{V}_0\right) + \phi + O(\Delta^2). \end{aligned} \quad (75)$$

According to Assumption 2, we can approximate $H^i((\Delta/\varepsilon), x, \mathbf{V}_0)$ by $H^i(\mathbf{V}_0)$, and obtain

$$\begin{aligned} V_\varepsilon^*(i, x) - V_0(i) &= -\rho\Delta V_0(i) + \Delta H^i(\mathbf{V}_0) + \phi + \phi' + O(\Delta^2) \end{aligned} \quad (76)$$

where $|\phi'| \leq \Delta(A/(\Delta/\varepsilon)^\alpha)$. Since, by definition, $\rho V_0(i) = H^i(\mathbf{V}_0)$, we readily obtain from (76) that

$$|V_\varepsilon^*(i, x) - V_0(i)| \leq \hat{E}(1 - \rho\Delta) + \Delta \frac{A}{\left(\frac{\Delta}{\varepsilon}\right)^\alpha} + O(\Delta^2). \quad (77)$$

This inequality holds for all $i \in I$ and $x \in X$. Therefore, one may replace the left-hand side with its supremum \hat{E} and obtain

$$\hat{E} \leq \hat{E}(1 - \rho\Delta) + \Delta \frac{A}{\left(\frac{\Delta}{\varepsilon}\right)^\alpha} + O(\Delta^2) \quad (78)$$

and, thus,

$$\hat{E} \leq \frac{1}{\rho} \left\{ \frac{A}{\left(\frac{\Delta}{\varepsilon}\right)^\alpha} + \frac{O(\Delta^2)}{\Delta} \right\}. \quad (79)$$

Taking again $\Delta(\varepsilon) = \varepsilon^{\alpha/(1+\alpha)}$, one obtains the result. •

C. Near Optimal Controls

Let $\tilde{u}_{\mathbf{v}}^i(\tau) \in U^i$ be as in (49)–(51), and let $\mathbf{V}_0 \in \Omega$ be the solution of (69). Denote by $V_{\varepsilon}^{\hat{u}}(i, x)$ the discounted expected cost associated with the implementation of the control

$$\hat{u}(\xi, t) = \tilde{u}_{\mathbf{V}_0}^{\xi} \left(\frac{t}{\varepsilon} \right), \quad \xi \in I. \quad (80)$$

Theorem 4: Let Assumptions of Theorem 3 be satisfied. Then, the following inequality holds:

$$|V_{\varepsilon}^{\hat{u}}(i, x) - V_{\varepsilon}^*(i, x)| \leq C\varepsilon^{\alpha/(1+\alpha)}. \quad (81)$$

The proof of this theorem is an immediate adaptation of the proof of Theorem 2.

VI. TURNPIKES AND DECOMPOSITION PRINCIPLE

A. Near Optimal Steady-State Controls

The limit control problem is particularly simplified in the case where (23)–(26) allow an asymptotically optimal steady-state solution. That is, if Assumption 4 is true.

Assumption 4: For each $i \in I$ and each $\mathbf{v} \in \Omega$, (23)–(26) allow an asymptotically optimal steady-state solution. That is,

$$H^i(\mathbf{v}) = L^i(\bar{x}_{\mathbf{v}}^i, \bar{u}_{\mathbf{v}}^i) + \sum_{j \in I} q_{ij}(\bar{x}_{\mathbf{v}}^i, \bar{u}_{\mathbf{v}}^i) v(j) \quad (82)$$

where $\bar{x}_{\mathbf{v}}^i \in X$ and $\bar{u}_{\mathbf{v}}^i \in U^i$ are defined as a solution of the problem

$$\min_{x \in X, u \in U^i} \left\{ L^i(x, u) + \sum_{j \in I} q_{ij}(x, u) v(j) \right\} \quad (83)$$

s.t.

$$0 = f^i(x, u) \quad (84)$$

$$u \in U^i. \quad (85)$$

Moreover,

$$\left| \frac{1}{\theta} \int_0^{\theta} h_{\mathbf{v}}^i(\tilde{x}(\tau), \bar{u}_{\mathbf{v}}^i) d\tau - H^i(\theta, x^0, \mathbf{v}) \right| \leq \frac{A}{\theta^{\alpha}} \quad (86)$$

where A and α are as in Assumption 2, and $\tilde{x}(\tau)$ is the solution of

$$\frac{d\tilde{x}(\tau)}{d\tau} = f^i(\tilde{x}(\tau), \bar{u}_{\mathbf{v}}^i) \quad (87)$$

$$\tilde{x}(0) = x^0. \quad (88)$$

Under Assumption 4, the control (80) does not explicitly depend on time, and has the form

$$\hat{u}(\xi) = \bar{u}_{\mathbf{V}_0}^{\xi}, \quad \xi \in I$$

Remark 7: Assumption 4 is satisfied under the following natural conditions (see, e.g., [26] and [27]):

- 1) for each $i \in I$, $f^i(x, u)$ is linear in x and u . That is, it has the form (29), with eigenvalues of A^i having negative real parts;

- 2) for each $i \in I$, U^i is convex and compact;
- 3) for any $v \in \Omega$, $i \in I$ the function

$$h_{\mathbf{v}}^i(x, u) \stackrel{\text{def}}{=} L^i(x, u) + \sum_{j \in I} q_{ij}(x, u) v(j)$$

is convex in (u, x) .

Assumption 4 postulates a kind of weak turnpike property. The name *turnpike* has been coined by economists when they applied the optimal control formalism to the optimal economic growth problems (see [65]). For a review of the conditions under which such a property holds, we refer to the book [18] (see also [36] and [37]). Let conditions 1), 2), and 3) from Remark 7 implying the fulfillment of Assumption 4 be satisfied. We introduce now the *upper level* controlled Markov chain defined as follows:

- the “action sets” are given by $\mathcal{A}^i = \{\bar{a} = (\bar{x}, \bar{u}) \in X \times U^i: 0 = f^i(\bar{x}, \bar{u})\}$ for each $i \in I$;
- the state set is I ;
- the cost rate is defined by $\mathcal{L}^i(\bar{a}) = L^i(\bar{x}, \bar{u})$ in state $i \in I$ and action $\bar{a} \in \mathcal{A}^i$;
- the transition rates are given by $\mathcal{Q}_{ij}(\bar{a}) = q_{ij}(\bar{x}, \bar{u})$.

Corollary 2: Under Assumptions 1, 2, and 4, the coupled differential equations (31) and (32) defining the *limit value function* in the finite-time horizon case, as well as the algebraic equation (69) in the infinite-horizon discounted-cost case, admit a solution which corresponds to the solution of the dynamic programming equations for the optimally controlled *upper level* Markov chain.

The proof is a direct verification.

This corollary permits us to give, in the infinite-horizon case, the following interpretation of the limit control problem as a decomposition scheme for the perturbed stochastic control problem:

Let $m = |I|$. Consider a set of $m + 1$ agents controlling the system. Each agent $i = 1, \dots, m$ controls the fast system when the discrete mode is $i \in I$. Hence, the agents are in one to one correspondence with the discrete modes. Agent 0 is a *coordinator*. The coordinator solves the *upper level* controlled Markov chain problem and sends to each agent $i = 1, \dots, m$ the optimal limit value vector $\mathbf{V}_0 = \{V_0(j): j \in I\}$. Now, given this information, agent i constructs an auxiliary cost rate

$$h_{\mathbf{V}_0}^i(x, u) = L^i(x, u) + \sum_{j \in I} q_{ij}(x, u) V_0(j)$$

and pilots the system, when it is in operational mode i , as if it were a deterministic control problem, with an infinite time horizon and an average cost criterion. As soon as the system jumps to state k , agent k constructs $h_{\mathbf{V}_0}^k(x, u)$ and proceeds in similar manner, and so on.

In the finite-horizon case, a similar, although more involved, interpretation could be developed:

With the same setting of $m + 1$ agents as above, the coordinator will send an information in the form of a limit value function $\mathbf{V}_0(t) = \{V_0(t, j): j \in I\}$, $t \in [0, T]$, obtained from the solution of the *upper level* controlled Markov chain problem on the time horizon $[0, T]$. Then, at each instant $\bar{t} \in [0, t]$, the agent $i \in I$ would have to solve an *infinitesimal* control problem

which, in the stretched out time scale, would also correspond to an infinite horizon deterministic control problem with cost rate

$$L^i(x, u) + \sum_{j \in I} q_{ij}(x, u) J_0(\bar{i}, j).$$

Note that a decomposition principle related to the one discussed above also appears in papers dealing with singularly perturbed Markov decision processes (see, e.g., [1]–[3], [12], [20], [50] and [51]).

B. Example

We conclude this section with an example in which all these assumptions are satisfied. Let

$$I = \{1, 2\} \quad m = p = 1 \quad U^1 = U^2 = [0, 1]$$

$$\begin{aligned} f^1(x, u) &= -x + u, & f^2(x, u) &= -2x + u \\ L^1(x, u) &= x^2 + u^2, & L^2(x, u) &= (x - \tfrac{1}{2})^2 + u^2 \\ q_{11}(u) &= -\tfrac{u}{2}, & q_{12}(u) &= \tfrac{u}{2}, & q_{21}(u) &= \tfrac{1}{2}, & q_{22}(u) &= -\tfrac{1}{2}. \end{aligned}$$

Notice that the problem so defined satisfies Assumption 1 with $X = [0, 1]$. It also satisfies Assumption 2 and 4 (see Remarks 1 and 7). Let us verify that (69) has a solution (notice that from Theorem 3 it follows that this solution can only be unique). For $i = 1$, (83)–(85) have the form

$$\min \left\{ x^2 + u^2 - \frac{u}{2}(v_1 - v_2) \right\} \quad (89)$$

s.t.

$$0 = -x + u, \quad u \in [0, 1]. \quad (90)$$

Their solution is

$$\tilde{x}_{\mathbf{v}}^1 = \bar{u}_{\mathbf{v}}^1 = \begin{cases} \frac{v_1 - v_2}{8}, & \text{if } \frac{v_1 - v_2}{8} \in (0, 1) \\ 0, & \text{if } \frac{v_1 - v_2}{8} \leq 0 \\ 1, & \text{if } \frac{v_1 - v_2}{8} \geq 1 \end{cases}. \quad (91)$$

By (82)

$$H^1(\mathbf{v}) = \begin{cases} -\frac{(v_1 - v_2)^2}{32}, & \text{if } \frac{v_1 - v_2}{8} \in (0, 1) \\ 0, & \text{if } \frac{v_1 - v_2}{8} \leq 0 \\ 2 - \frac{v_1 - v_2}{2}, & \text{if } \frac{v_1 - v_2}{8} \geq 1 \end{cases}. \quad (92)$$

For $i = 2$, (83)–(85) are of the form

$$\min \left\{ (x - \tfrac{1}{2})^2 + u^2 + \tfrac{1}{2}(v_1 - v_2) \right\} \quad (93)$$

s.t.

$$-2x + u = 0, \quad u \in [0, 1]. \quad (94)$$

Their solution is

$$\tilde{x}_{\mathbf{v}}^2 = \bar{u}_{\mathbf{v}}^2 = \tfrac{1}{5} \quad (95)$$

and

$$H^2(\mathbf{v}) = \tfrac{1}{5} + \tfrac{1}{2}(v_1 - v_2). \quad (96)$$

Substituting (92) and (96) in (69), one can verify that it allows the only solution

$$v_1 = 0, \quad v_2 = \frac{1}{5(\rho + \frac{1}{2})}. \quad (97)$$

Hence, the conditions of Theorem 4 are satisfied and, by (91) and (95), near optimal control policy consists of using control $u = 0$ in mode 1 and control $u = 1/5$ in mode 2.

VII. CONCLUSION

We have proposed a new technique for the study of a class of singularly perturbed control systems. It uses the fundamental tenet of transition, characterizing the dynamic programming approach, in association with a particular averaging technique for the fast mode. This method is particularly well adapted to the case of PDCSSs, when the fast dynamics are associated with the deterministic part and the slow mode corresponds to the infrequent jump disturbance process.

We have shown that the procedure for finding a near optimal solution for the given class of problems can be divided into two parts: optimization of the fast dynamics for each operational mode separately (achieved via the solution of the infinitesimal problems) and the solution of the limit equation taking the form of ordinary differential equations for finite-time horizon and the form of algebraic equations for infinite-time horizon with discounting.

The decomposition principle interpretation obtained when the infinitesimal control system possesses the turnpike property, permits us to better understand the behavior of some real life processes, such as the manufacturing systems that have been modeled as PDCSSs. If the jump disturbances are modeled as a continuous-time Markov chain with constant jump rates, the limit control problem will yield, in each mode $i \in I$, an optimal control that would be unaffected by the \mathbf{J}_0 value vector. In such a case, in each mode $i \in I$, the optimal control will be approximately the one that would correspond to a purely deterministic system starting in the mode i and never switching to another mode. When the jump rates depend on the state and control, the situation is different and the upper-level limit control process becomes, when the turnpike property holds, a discrete-state compact-action Markov decision process.

Finally, we would like to emphasize that a number of important results were recently obtained in the theory of linear jump control systems (see [24], [45], [49], and [67], and the references therein). Most of these results were derived via an analysis of differential or algebraic Riccati-type equations which is possible only for systems with special “linear-quadratic” structure and unconstrained controls. In contrast to these results, our approach allows consideration of nonlinear systems and constrained controls. Also, it allows us to consider classes of systems with probabilities of jumps depending on controls. However, note that, unlike the works cited above (on linear jump control systems), we restrict ourselves in this paper to the case when the state of continuous-time system and that of the “jump” process are both observable at each moment of time. A question of applicability of our approach to a general situation is left

open.

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